Solution to

Midterm Examination

Answer all questions. You should justify your answer and show all details.

1. (10 points) Let T be a triangle formed by the lines $y = x + 6, y + 2x = 0$ and the y-axis. Evaluate
$$\iint_T y\,dA(x, y)\ .$$

2. (10 points) Let D be the region pinched between the circles $x^2+y^2 = 4$ and $x^2+(y-1)^2 = 1$. Find its centroid.

3. (10 points) Sketch the region of the following iterated integral
$$\int_{\pi/4}^{3\pi/4} \int_0^{6/(2\sin\theta-\cos\theta)} r^2 \cos\theta\, dr d\theta$$
and then evaluate it in $dydx$.

4. (10 points) Sketch the polar curve $r = 3\cos 4\theta$. How many leaves it has? Find the area of one of its leaf.

5. (10 points) Find the volume of the tetrahedron whose vertices are $(0, 0, 0), (1, 0, 0), (0, 3, 0)$ and $(0, 0, 5)$.

6. (10 points) Find the volume of the solid which is bounded below by the xy-plane, on the sides by the sphere $\rho = 3$, and above by the cone $\varphi = \pi/6$.

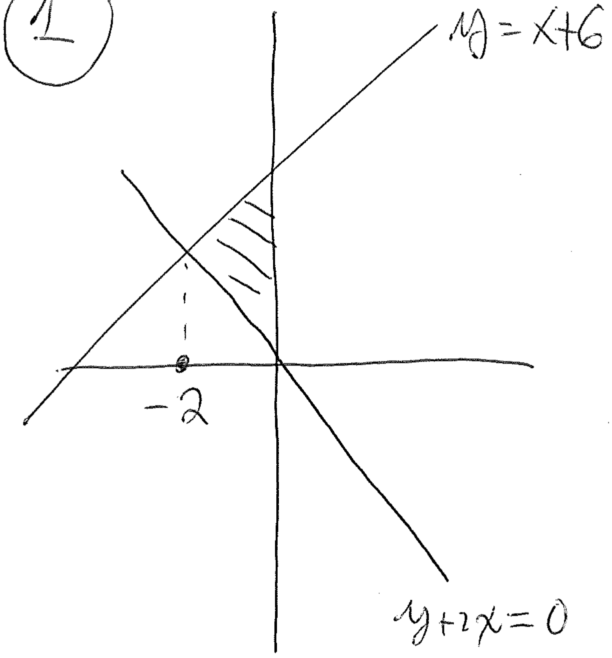
7. (15 points) Let C be the intersection of the ball $x^2 + y^2 + z^2 \leq 4$ and the solid cylinder $x^2 + y^2 \leq 3$. Express the integral
$$\iiint_C f(x, y, z)\, dV$$
respectively in cylindrical and spherical coordinates.

8. (a) (5 points) Let Ω be a region in space whose cross section $\Omega(z)$ is a two dimensional region for each $z \in [a, b]$. Suppose that Ω is the union of all these cross sections. Use Riemann sums to explain why the volume of Ω is equal to:
$$\int_a^b |\Omega(z)|\, dz\ ,$$
where $|\Omega(z)|$ is the area of $\Omega(z)$ and $\Omega(z)$ is $\{(x, y) : (x, y, z) \in \Omega\}$.
   (b) (10 points) Use this formula to find the volume enclosed by the ellipsoid $6x^2 + 6y^2 + z^2 = 1$.

9. (10 points) Let D be the region enclosed by the polar curve $\{(x, y) : 0 \leq r \leq r(\theta), 0 \leq \theta \leq 2\pi\}$. Suppose it is radially symmetric, that is, $r(\theta + \pi) = r(\theta)$. Show that for a thin object occupying D with density δ satisfying $\delta(-x, -y) = \delta(x, y)$, the center of mass of this object is $(0, 0)$.

Midterm Exam

(1)



$$\iint_T y\,dA = \int_{-2}^{0}\int_{-2x}^{x+6} y\,dy\,dx$$

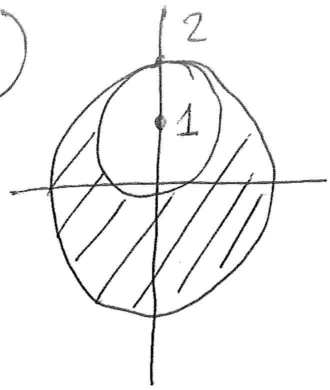
$$= \frac{1}{2}\int_{-2}^{0}\left[(x+6)^2-(-2x)^2\right]dx$$

$$= \frac{1}{2}\int_{-2}^{0}(-3x^2+12x+36)\,dx$$

$$\vdots$$

$$= 20 \ \#$$

(2)



$M =$ area of big disk − area of small disk
$= 4\pi - \pi$
$= 3\pi$

$\bar{x} = 0$ by symmetry.

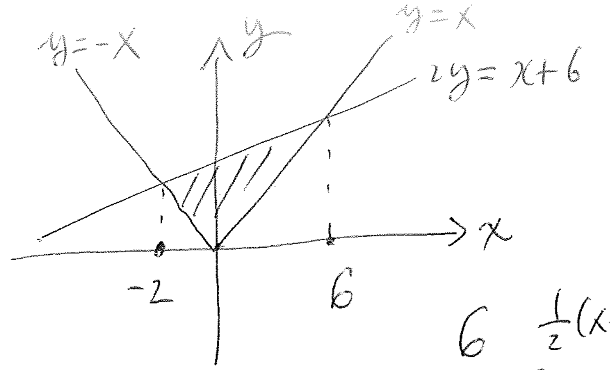
$$\bar{y} : \iint_D y\,dA = \iint_{D_{big}} y\,dA - \iint_{D_{small}} y\,dA$$
$$= 0 - \iint_{D_{small}} y\,dA$$

D_{small} : $x^2 + (y-1)^2 \leq 1$
$\Leftrightarrow r = 2\sin\theta,$
$\theta \in [0, \pi]$ $\therefore$

$$-\iint_{D_{small}} y\,dA = -\int_0^{\pi}\int_0^{2\sin\theta} r\sin\theta \; r\,dr\,d\theta$$
$$= -\int_0^{\pi} \frac{1}{3} r^3 \Big|_0^{2\sin\theta} \sin\theta\,d\theta$$
$$= -\frac{8}{3}\int_0^{\pi} \sin^4\theta\,d\theta$$
$$= -\frac{8}{3}\int_0^{\pi} \left(\frac{1}{2}(1-\cos 2\theta)\right)^2 d\theta$$
$$= -\frac{8}{3}\int_0^{\pi} \frac{1}{4}(1 - 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$
$$= -\frac{2}{3}\int_0^{\pi} \left(1 + \frac{1}{2}(1+\cos 4\theta)\right) d\theta$$
$$= -\frac{2}{3} \times \frac{3}{2} \times \pi = -\pi. \quad \bar{y} = -\pi/3\pi = -1/3.$$

$\therefore$ Centroid $= (0, \bar{y}) = (0, -1/3)$.

3



$$\text{The integral} = \int_0^6 \int_x^{\frac{1}{2}(x+6)} x \, dy \, dx + \int_{-2}^0 \int_{-x}^{\frac{1}{2}(x+6)} x \, dy \, dx$$

$$= \int_0^6 x\left(\tfrac{1}{2}(x+6) - x\right) dx + \int_{-2}^0 x\left(\tfrac{1}{2}(x+6) + x\right) dx$$

$$= \frac{1}{2}\int_0^6 (-x^2+6x)\, dx + \frac{1}{2}\int_{-2}^0 (3x^2+6x)\, dx$$

$$= \frac{1}{2}\left(-\frac{x^3}{3} + 3x^2\right)\Big|_0^6 + \frac{1}{2}\left(x^3+3x^2\right)\Big|_{-2}^0$$

$$= \frac{1}{2}(-72+108) + \frac{1}{2}(8-12)$$

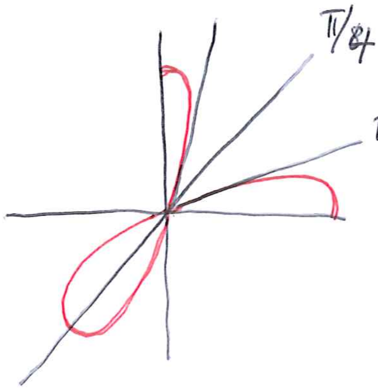
$$= 16.$$

(4) $r = 3\cos 4\theta$

$\cos\theta$ of period 2π

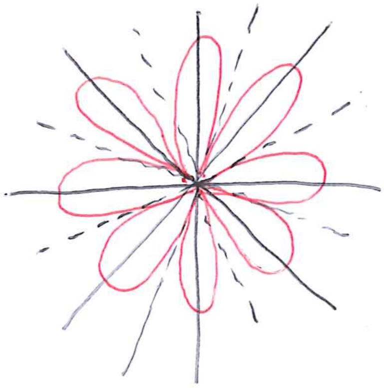
$\cos(4\theta)$ of period $\pi/2$

over $[0, \frac{\pi}{2}]$, the graph looks like :



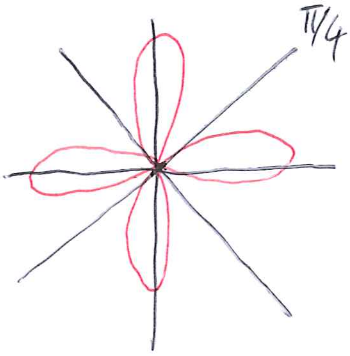
rotate it $90°$ each time to get

8 leaves.



(If you don't accept $r<0$,

4 leaves are fine.)



(The convention is : when $r<0$, draw in the opposite way)

Area of one leaf

$$= 2\int_0^{\pi/8}\int_0^{3\cos 4\theta} r\,dr\,d\theta = \int_0^{\pi/8} 9\cos^2 4\theta\,d\theta$$

$$= \frac{9}{2}\int_0^{\pi/8}(1+\cos 8\theta)\,d\theta$$

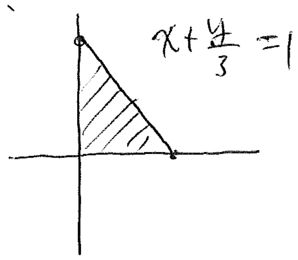
$$= \frac{9}{16}\pi \quad \#$$

(5) The plane passing through $(1,0,0)$, $(0,3,0)$, $(0,0,5)$

is $\quad x + \frac{y}{3} + \frac{z}{5} = 1$, ie

$$15x + 5y + 3z = 15$$

T :

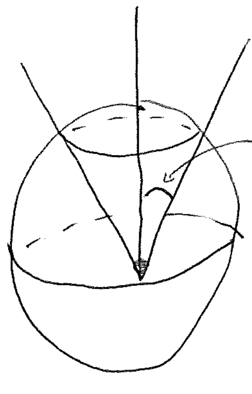


$$\text{vol} = 5\iint_T \left(1 - x - \frac{y}{3}\right) dA(x,y) ,$$

$$= \frac{1}{3}\int_0^1 \int_0^{3-3x} (15 - 15x - 5y)\, dy\, dx$$

$\vdots$

$$= 5/2$$



(6) volume

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{\rho} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$\vdots$

$=$

(some understood it as

$$\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\rho} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$= \cdots = \ldots$

Both are fine.)

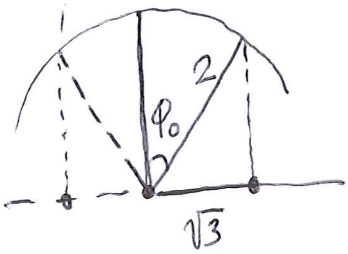
(7) $\Omega = \left\{ (x,y,z) : \begin{array}{l} -\sqrt{4-x^2-y^2} \le z \le \sqrt{4-x^2-y^2} \\ 0 \le x^2+y^2 \le 3 \end{array} \right\}$

$= \left\{ (x,y,z) : \begin{array}{l} -\sqrt{4-r^2} \le z \le \sqrt{4-r^2} \\ 0 \le r \le \sqrt{3}, \\ 0 \le \theta \le 2\pi \end{array} \right\}$

In cylindrical coordinate,

$$\iiint_\Omega f\,dV = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta .$$

In spherical coordinate,



$$\iiint_\Omega f\,dV = \int_0^{2\pi} \int_0^{\pi/3} \int_0^{2} \hat{f}(\rho,\varphi,\theta)\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$+ \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_0^{\sqrt{3}/\sin\varphi} \hat{f}(\rho,\varphi,\theta)\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$\int_0^{2\pi} \int_{2\pi/3}^{\pi} \int_0^{2} \hat{f}(\rho,\varphi,\theta)\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta ,$$

$\sin\varphi_0 = \sqrt{3}/2$

$\varphi_0 = \pi/3$

cylinder

$x^2+y^2=3$

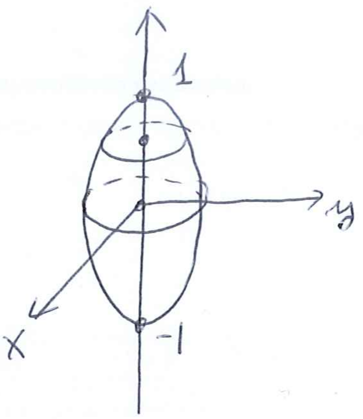
$\rho = \sqrt{3}/\sin\varphi$

where $\hat{f}(\rho,\varphi,\theta) = f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$.

Note: Some forgot to put in f, some wrote $f(\rho,\varphi,\theta)$, both no good. Writing $f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)$ is fine.

⑧ (a) see Notes

(b) For $z \in (-1, 1)$, $\Omega(z)$ is bounded by $x^2+y^2 = \frac{1}{6}(1-z^2)$.



$\Omega(z)$ is a disk of radius $\frac{1}{6}(1-z^2)$.

$\therefore \quad |\Omega(z)| = \pi \frac{1}{6}(1-z^2)$

$$\text{volum of ellipsoid} = \int_{-1}^{1} \frac{\pi}{6}(1-z^2)\,dz$$

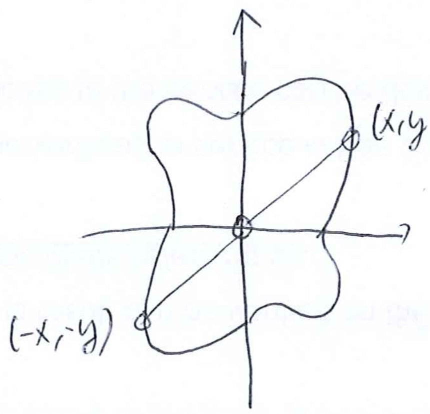
$$= 2\int_0^1 \frac{\pi}{6}(1-z^2)\,dz$$

$$= \frac{1}{3}\pi\left(z - \frac{z^3}{3}\right)\Big|_0^1$$

$$= \frac{1}{3}\pi \times \frac{2}{3}$$

$$= \frac{9\pi}{2} \ \#$$

⑨



$$M_x = \iint_D y\, dA(x,y)$$

$$= \int_0^{2\pi}\int_0^{r(\theta)} r\sin\theta$$

$$= \int_0^{\pi}\int_0^{r(\theta)} r\sin\theta \cdot$$

$$+ \int_{\pi}^{2\pi}\int_0^{r(\theta)} r\sin\theta \qquad \text{— ①}$$

Now $\displaystyle\int_{\pi}^{2\pi}\int_0^{r(\theta)} r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi}\int_0^{r(\alpha+\pi)} r^2 \sin(\alpha+\pi)\,dr\,d\alpha$

$\alpha = \theta - \pi$

$d\alpha = d\theta$

$$M_x = \iint_D y\,\delta(x,y)\,dA(x,y)$$

$$= \int_0^{2\pi}\int_0^{r(\theta)} r\sin\theta\,\delta(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$= \left(\int_0^{\pi}\int_0^{r(\theta)} + \int_{\pi}^{2\pi}\int_0^{r(\theta)}\right)\left(r^2\sin\theta\,\delta(r\cos\theta, r\sin\theta)\right)dr\,d\theta \qquad \text{—— (1)}$$

$$\int_{\pi}^{2\pi}\int_0^{r(\theta)} r^2\sin\theta\,\delta(r\cos\theta, r\sin\theta)\,dr\,d\theta = \int_0^{\pi}\int_0^{r(\alpha+\pi)} r^2\sin(\alpha+\pi)\,\delta(r\cos(\alpha+\pi), r\sin(\alpha+\pi))\,dr\,d\alpha$$

$$= \int_0^{\pi}\int_0^{r(\alpha)} -r^2\sin\alpha\,\delta(-r\cos\alpha, -r\sin\alpha)\,dr\,d\alpha$$

$\theta = \alpha+\pi$

$d\theta = d\alpha$

(use $r(\alpha+\pi) = r(\alpha)$, $\delta(-x,-y) = \delta(x,y)$)

$$= -\int_0^{\pi}\int_0^{r(\alpha)} r^2\sin\alpha\,\delta(r\cos\alpha, r\sin\alpha)\,dr\,d\alpha$$

$$= -\int_0^{\pi}\int_0^{r(\theta)} r^2\sin\theta\,\delta(r\cos\theta, r\sin\theta)\,dr\,d\theta$$

(change notation from α to θ)

Put in (1):

$$M_x = \left(\int_0^{\pi}\int_0^{r(\theta)} - \int_0^{\pi}\int_0^{r(\theta)}\right)(\ldots)\,dr\,d\theta = 0.$$

$\therefore\ \bar{y} = 0.$

Similarly, $\bar{x} = 0$.